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Generalized exclusion statistics and degenerate signature of strongly interacting anyons

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We show that below the degenerate temperature the distribution profiles of strongly interacting anyons in one dimension coincide with the most probable distributions of ideal particles obeying generalized exclusion statistics (GES). In the strongly interacting regime the thermodynamics and the local two-particle correlation function derived from the GES are seen to agree for low temperatures with the results derived for the anyon model using the thermodynamic Bethe Ansatz. The anyonic and dynamical interactions implement a continuous range of GES, providing a signature of strongly interacting anyons, including the strongly interacting one-dimensional Bose gas.

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I. INTRODUCTION

A fundamental principle of quantum statistical mechanics is the existence of two types of particles – bosons and fermions – obeying Bose-Einstein and Fermi-Dirac statistics. The Pauli exclusion principle dictates that no two fermions can occupy the same quantum state. There is no such restriction on bosons. These rules lead to a number of striking and subtle quantum many-body effects. In three dimensions particles are either bosons or fermions. However, in two-dimensions, anyons [1] may also exist, obeying fractional statistics. Fractional statistics have recently been observed in experiments on the elementary excitations of a two-dimensional electron gas in the fractional quantum Hall (FQH) regime [2], where the quasi-particles are charged anyons. The concept of anyons has become important in studying the FQH effect [2, 3, 4] and in topological quantum computation [4, 5].

A more general description of quantum statistics is provided by Haldane exclusion statistics [6]. Haldane formulated a description of fractional statistics based on a generalized Pauli exclusion principle, which is now called generalized exclusion statistics (GES). The definition of GES is independent of space dimension. A key point of GES is to count the dimensionality of the single-particle Hilbert space as extra particles are added. It is of interest to find applications of Haldane GES in one-dimensional (1D) and two-dimensional quantum systems. A particular property of 1D many-body interacting systems is that pairwise dynamical interaction between identical particles is inextricably related to their statistical interaction. Anyons in one dimension acquire a step-function-like phase when two identical particles exchange their positions in the scattering process. It follows that 1D topological effects and dynamical interaction are not separable. This transmutation between dynamical interaction and statistical interaction has been observed in explicit calculations on the 1D Calogero-Sutherland model [7] and the 1D interacting Bose gas [8, 9], which are equivalent to an ideal gas obeying GES. The 1D interacting model of anyons proposed by Kundu [10] provides further evidence of the transmutation between dynamical and statistical interaction [11]. This integrable anyon model reduces to the 1D interacting Bose gas [12] and the free Fermi gas as special cases. Very recently an anyon-fermion mapping has been proposed to obtain solutions for several models of ultracold gases with 1D anyonic exchange symmetry [13].

The experimental realization of the 1D Tonks-Girardeau (TG) gas [14, 15, 16], has provided significant insights into the fermionization of interacting Bose systems. The pure TG gas is obtained in the infinite interaction strength limit of the 1D Bose gas [12]. The experimental measurement of local pair correlations in the strongly interacting 1D Bose gas reveals an important feature, namely that the wave functions overlap even in the TG regime [17]. This implies that the strongly interacting 1D Bose gas does not exhibit pure Fermi behaviour. Equivalently, the statistics are not strictly Fermi-Dirac. A deviation from Fermi-Dirac statistics also appears in the strongly interacting limit of the anyon gas [11], also called the anyonic TG gas [13].

In this paper we give a quantitative description of the distribution profiles of the strongly interacting 1D anyon gas, including the strongly interacting Bose gas and the TG limit. We show that these profiles are equivalent to the most probable distribution profiles of ideal particles obeying nonmutual GES at temperatures below a degenerate temperature T_d. This equivalence between strongly interacting anyons and ideal particles with nonmutual GES is further supported by the thermodynamic properties and the local two-body correlation functions, which we derive independently from the thermodynamic Bethe Ansatz (TBA) equations for the anyon model and via GES [18, 19, 20]. We conclude that 1D interacting anyons and bosons in the strong coupling regime are properly described by GES. The quasiparticle excitations for these systems obey GES. The anyonic interaction and the dynamical interaction implement a continuous range of GES. At low temperatures, i.e., $k_BT < 0.1T_d$, the dynamical interaction and anyonic statistical interaction determine the GES and the thermal fluctuations are sup-

pressed. At zero temperature they recover the properties derived from the Bethe ansatz equations (BAE) [11]. In addition, the distribution profiles indicate that the dynamical interaction and anyonic statistical interaction may implement a duality between the GES parameter α and its inverse $1/\alpha$.

This paper is set out as follows. In section II, we present the Bethe Ansatz solution of the 1D interacting anyon model. The ground state properties are also analysed. We demonstrate the quivalence between 1D interacting anyons and ideal particles obeying GES in section III. Distribution profiles of 1D interacting anyons and ideal particles obeying GES are discussed in section IV. In section V, we compare the thermodynamics and the local pair correlation functions obtained via the TBA and GES approaches. Section VI is devoted to concluding remarks and a brief discussion.

II. THE MODEL

We consider the 1D interacting anyon model with hamiltonian [10]

$$H_N = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + g_{1\mathrm{D}}\sum_{1\le i<j\le N}\delta(x_i - x_j) \qquad (1)$$

under periodic boundary conditions. Here m denotes the atomic mass. In analogy with the 1D confined Bose gas [21], the coupling constant is determined by $g_{1\mathrm{D}} = \hbar^2 c/m = -2\hbar^2/(ma_{1\mathrm{D}})$ where the coupling strength c is tuned through an effective 1D scattering length $a_{1\mathrm{D}}$ via confinement. Hereafter we set $\hbar = 2m = 1$ for convenience. We shall restore physical units in the thermodynamics later. The ground state properties and Haldane exclusion statistics for the model (1) have been studied recently in Ref. [11]. The hamiltonian (1) exhibits both anyonic statistical and dynamical interactions which result in a richer range of quantum effects than those of the 1D interacting Bose gas [12].

The Bethe Ansatz wave function is written as [10, 11]

$$\begin{aligned}\chi(x_1\ldots x_N) \;&=\; \mathrm{e}^{-\frac{\mathrm{i}\kappa}{2}\sum_{x_i<x_j}^{N} w(x_i,x_j)}\sum_P A(k_{P1}\ldots k_{PN})\\ &\quad\times \mathrm{e}^{\mathrm{i}(k_{P1}x_1+\cdots+k_{PN}x_N)}. \end{aligned}\qquad (2)$$

It satisfies the anyonic symmetry

$$\chi(\ldots x_i\ldots x_j\ldots) = \mathrm{e}^{-\mathrm{i}\theta}\chi(\ldots x_j\ldots x_i\ldots) \qquad (3)$$

with the anyonic phase

$$\theta = \kappa\left(\sum_{k=i+1}^{j} w(x_i,x_k) - \sum_{k=i+1}^{j-1} w(x_j,x_k)\right) \qquad (4)$$

for $i<j$. In equation (2), the summation is over all $N!$ permutations P. The coefficents $A(k_{P1}\ldots k_{PN})$ are determined by the two-body scattering relation [10, 11]

$$A(\ldots k_j,k_i\ldots) = \frac{k_j - k_i + \mathrm{i}c'}{k_j - k_i - \mathrm{i}c'}A(\ldots k_i,k_j\ldots). \qquad (5)$$

In the above equations the multi-step function $w(x_1,x_2) = -w(x_2,x_1) = 1$ for $x_1 > x_2$, with $w(x,x) = 0$. κ is the anyonic interaction parameter. $\{x_i\}$ are the coordinates of particles in length L. The anyonic parameter κ is the key difference from the standard 1D Bose gas [12], for which $\kappa = 0$. The interacting anyons in the strong coupling regime may be viewed as the anyonic TG gas with zero range interaction. This has been verified using anyon-fermion mapping for the hard core anyons [13].

The energy eigenvalues are given by $E = \sum_{j=1}^{N} k_j^2$, where the individual quasimomenta k_j satisfy [10, 11]

$$\mathrm{e}^{\mathrm{i}k_jL} = -\mathrm{e}^{\mathrm{i}\kappa(N-1)}\prod_{\ell=1}^{N}\frac{k_j - k_\ell + \mathrm{i}\,c'}{k_j - k_\ell - \mathrm{i}\,c'} \qquad (6)$$

for $j = 1,\ldots,N$. We will refer to these equations as the BAE. The anyonic parameter κ and the dynamical interaction c are inextricably related via the effective coupling constant

$$c' = c/\cos(\kappa/2). \qquad (7)$$

We use a dimensionless coupling constant $\gamma = c/n$ to characterize different physical regimes of the anyon gas, where $n = N/L$ is the linear density. In this paper, we consider the TG regime where $\gamma \gg 1$ with $0 \le \kappa < 4\pi$. The model reduces to the interacting Bose gas [12] at $\kappa = 0$. For $\kappa = \pi$ and 3π it reduces to the free Fermi gas. The anyonic TG gas lies in the range $0 \le \kappa \le \pi$ and $3\pi \le \kappa \le 4\pi$, where the effective interaction $c' > 0$. However, if the anyon parameter κ is tuned smoothly from $\kappa < \pi$ to $\kappa > \pi$, i.e., $\pi \le \kappa \le 3\pi$, the effective interaction is attractive. Here the strong Fermi pressure may prevent the anyons from collapsing. We call this the super anyonic TG gas.

In general the global phase factor behaves like a topological effect associated with Aharonov-Bohm and Aharonov-Casher fluxes in a 1D mesoscopic ring [22]. The parameter κ leads to a shift in quasimomenta $k_i \to k_i + \kappa(N-1)/L$ which gives rise to excited states. Due to the periodicity of the BAE, we only consider an effect from $\kappa(N-1) = \nu \pmod{2\pi}$ with $-\pi \le \nu \le \pi$ [11]. In the thermodynamic limit, the ground state is given by $E = N(n^2 e(\gamma,\kappa) + \nu^2/L^2)$ where $e(\gamma,\kappa) = \frac{\gamma^3}{\lambda^3}\int_{-1}^{1} g(x)x^2dx$. The root density distribution $g(x)$ and the parameter $\lambda = c/Q$ are determined by Lieb-Liniger type integral equations of the form

$$\begin{aligned} g(x) \;&=\; \frac{1}{2\pi} + \frac{\lambda\cos(\kappa/2)}{\pi}\int_{-1}^{1}\frac{g(y)dy}{\lambda^2 + \cos^2(\frac{\kappa}{2})(x-y)^2}\\ \lambda \;&=\; \gamma\int_{-1}^{1} g(x)dx. \end{aligned}\qquad (8)$$

In the above Q is the cut-off momentum. For the strong coupling regime the lowest energy per particle is given by

$$\frac{E_0}{N} \approx \frac{\hbar^2}{2m}\frac{\pi^2}{3}n^2\left(1-4\gamma^{-1}\cos(\kappa/2)\right)+\nu^2/L^2. \qquad (9)$$

In the thermodynamic limit, the last term can be ignored compared to the kinetic energy. At zero temperature the ground state energy E_0 can be identified with that for hard-rod anyons with an effective length $L_{\rm eff} = L(1-na_{1D}\cos(\kappa/2))$ for $\gamma = -2/(na_{1D}) \gg 1$. Here a_{1D} is negative. The effective length becomes smaller than L. This causes the linear density n to be increased, thus the kinetic energy, being proportional to n^2, also increases. However, if $0<\kappa<\pi$ or $3\pi<\kappa<4\pi$, the effective length becomes larger than L due to repulsion. This leads to a smaller kinetic energy than the free Fermi energy. In this way the anyonic paramter κ triggers a resonance-like behaviour from strongly repulsive interaction to strongly attractive interaction. In the resonance region, i.e., $na_{1D}\cos(\kappa/2))\approx 0$, the TG anyon gas and the super TG gas correspond to the GES duality between α and its inverse $1/\alpha$. We define the GES parameter α and discuss this behaviour further below.

III. GES DESCRIPTION

According to Haldane exclusion statistics, the GES parameter α_{ij} is defined through the linear relation $\Delta d_i/\Delta N_j = -\alpha_{ij}$ [6], i.e., the number of available single particle states of species i, denoted by d_i, depends on the number of other species $\{N_j\}$ when one particle of species i is added. Thus d_i is given by [8, 9]

$$d_i(\{N_j\}) = G_i^0 - \sum_j \alpha_{ij}N_j. \qquad (10)$$

Here $G_i^0 = d(\{0\})$ is the number of available single particle states when no particles present in the system. The number of configurations at fixed $\{N_j\}$ when one particle of the ith species is excited is [8]

$$W(\{N_i\}) = \prod_i \frac{(d_i+N_i-1)!}{(N_i)!(d_i-1)!} \qquad (11)$$

which gives the grand partition function

$$Z = \sum_{\{N_i\}} W(\{N_i\}) \mathrm{e}^{\sum_i N_i(\mu_i-\epsilon_i)/K_BT}. \qquad (12)$$

In deriving the quantum statistics [8], the most probable distribution is given by $\sum_j(w_j\delta_{ij}+\beta_{ij})n_j = 1$, where w_i is determined by

$$(1+w_i)\prod_j\left(\frac{w_j}{1+w_j}\right)^{\alpha_{ji}} = \mathrm{e}^{(\epsilon_i-\mu_i)/K_BT}. \qquad (13)$$

In the above, $n_i = N_i/G_i$ and $\beta_{ij} = \alpha_{ij}G_j/G_i$ as defined in Ref. [8]. For $\alpha = 0$ the GES result (13) reduces to Bose-Einstein statistics with $n_i = 1/(\mathrm{e}^{(\epsilon_i-\mu)/K_BT}-1)$. For $\alpha_{ij} = 1$ it reduces to Fermi-Dirac statistics with $n_i = 1/(\mathrm{e}^{(\epsilon_i-\mu)/K_BT}+1)$.

On setting $w_i = \mathrm{e}^{\epsilon(k_i)/K_BT}$ we see that equation (13) becomes

$$\epsilon(k_i) = \epsilon_i - \mu_i - K_BT\sum_j(\delta_{ji}-\alpha_{ji})\left(1+\mathrm{e}^{-\frac{\epsilon(k_j)}{K_BT}}\right) \qquad (14)$$

which is identical to the TBA equations

$$\epsilon(k) = \epsilon - \mu - \frac{K_BT}{2\pi}\int_{-\infty}^{\infty} dk' \theta'(k-k')\ln(1+\mathrm{e}^{-\frac{\epsilon(k')}{K_BT}}) \qquad (15)$$

for the anyon model subject to the identification

$$\alpha_{ij} := \alpha(k,k') = \delta(k,k') - \frac{1}{2\pi}\theta'(k-k') \qquad (16)$$

with a dispersion relation $\epsilon = \frac{\hbar^2}{2m}k^2$. This definition allows different species to refer to identical particles with different quantum numbers or with different momenta. Here the function

$$\theta'(x) = \frac{2c\cos(\kappa/2)}{c^2+\cos^2(\kappa/2)x^2}. \qquad (17)$$

We note that this type of connection between TBA and GES appears to be very general for 1D interacting systems [23], including multicomponent systems [24].

For the TBA, quasiparticle excitations are characterized by the density of occupied states and the density of holes denoted by ρ and $\rho_{\rm h}$, repectively. The dressed energy is expressed by $\rho_{\rm h}/\rho = \mathrm{e}^{\frac{\epsilon(k)}{K_BT}}$. In the ground state $\rho_{\rm h} = 0$. At arbitrary temperatures, we have the BAE

$$\rho(k) + \rho_{\rm h}(k) = \frac{1}{2\pi} + \frac{1}{2\pi}\int_{-\infty}^{\infty} dk'\theta'(k-k')\rho(k') \qquad (18)$$

which can give rise in the strongly interacting limit to a relation between the particle distribution $\rho(k)$ and the hole distribution $\rho_{\rm h}(k)$ of the form

$$2\pi\left(\rho(k)+\rho_{\rm h}(k)\right) \approx 1/\alpha \qquad (19)$$

with [11]

$$\alpha = 1 - 2\gamma^{-1}\cos(\kappa/2). \qquad (20)$$

This relation clearly indicates the nature of the deviation from Fermi-Dirac statistics. At low temperatures $\rho_{\rm h} \ll \rho$. It follows that, up to the negligible quantity $2\cos(\kappa/2)\rho_{\rm h}/(\gamma\rho)$, we have $2\pi\left(\alpha\rho+\rho_{\rm h}\right)\approx 1$. This relation indicates that at low temperatures a one particle excitation leaves α holes below the Fermi surface. Minimizing the free energy [7]

$$\frac{F}{L} = \int_{-\infty}^{\infty} dk\rho\left(\epsilon-\mu\right) \qquad (21)$$
$$-K_BT\int_{-\infty}^{\infty} dk\left\{(\rho+\rho_{\rm h})\ln(\rho+\rho_{\rm h}) - \rho\ln\rho - \rho_{\rm h}\ln\rho_{\rm h}\right\}$$

with this relation gives the most probable distribution $n(\epsilon) = 2\pi\rho$ to be

$$n(\epsilon) = \frac{1}{\alpha + w(\epsilon)} \tag{22}$$

where $w(\epsilon)$ satisfies the equation

$$w^{\alpha}(\epsilon)\left(1 + w(\epsilon)\right)^{1-\alpha} = \mathrm{e}^{\frac{\epsilon-\mu}{K_B T}}. \tag{23}$$

This is precisely equation (13) with nonmutual statistics, i.e., with $\alpha_{ij} = \alpha\delta_{ij}$.

We have thus shown that strongly interacting anyons map to an ideal gas obeying nonmutual GES α at low temperatures. We derive the statistics from (22) below. It is clear to see that at zero temperature there is a Fermi-like surface with a cut-off energy $\epsilon_{\mathrm{F}} = \mu_0$, where $n(k) = 1/\alpha$ if $\epsilon(k) \le \mu_0$ and $n(k) = 0$ if $\epsilon(k) > \mu_0$. At low temperatures, a relatively small number of particles are excited above the Fermi surface leaving an unequal amount of holes below the Fermi surface due to the collective effect. At high temperatures, there are a large number of holes which drive the system towards a Maxwell-Boltzman distribution of particles.

IV. DISTRIBUTION PROFILES

A quantitative explanation of the GES induced by the dynamical interaction γ and the anyonic statistical interaction κ follows from the calculation of the statistics and the thermodynamics via either the TBA equation (15) or the GES results (22) and (23). To this end, we express the particle number and total energy as

$$N = \int_0^{\infty} d\epsilon\, G(\epsilon)n(\epsilon), \quad E = \int_0^{\infty} d\epsilon\, G(\epsilon)n(\epsilon)\epsilon \tag{24}$$

where $n(\epsilon)$ is determined from (22). The pressure is given by the relation $PL = 2E$ and the density of states by

$$G(\epsilon) = L/\left(2\pi\sqrt{\frac{\hbar^2}{2m}\epsilon}\right). \tag{25}$$

In the thermodynamic limit we may ignore the factor ν/L, as it is much smaller than the cut-off quasimomentum.

In order to calculate the thermodynamic properties at low temperatures, we bring them into line with the Sommerfeld expansion. For convenience, we make the change $w(\epsilon) \to 1/w(\epsilon) - 1$ in equations (22) and (23), giving

$$n(\epsilon) = \frac{w(\epsilon)}{1 + (\alpha - 1)w(\epsilon)}, \tag{26}$$

$$\alpha \ln\left(1 - w(\epsilon)\right) - \ln w(\epsilon) = \frac{\epsilon - \mu}{K_B T}. \tag{27}$$

We find that $w(\infty) \approx 0$ and

$$w(0) \approx 1 - \mathrm{e}^{-\frac{\mu}{\alpha K_B T}} - \frac{1}{\alpha}\mathrm{e}^{-\frac{2\mu}{\alpha K_B T}} + O(\mathrm{e}^{-\frac{3\mu}{\alpha K_B T}}). \tag{28}$$

Following Isakov *et al.* [18], at low temperatures, i.e., for $T < T_d$, where $T_d = \frac{\hbar^2}{2m}n^2$ is the quantum degeneracy temperature, the thermodynamics of ideal particles obeying GES can be derived from the fractional statistics $n(\epsilon)$ (26) via Sommerfeld expansion. Let us define $I[f] = \int_0^{\infty} d\epsilon f(\epsilon)n(\epsilon)$. This integral can be calculated explicitly with the help of (26) and (27). It follows that

$$I[f] = \frac{1}{\alpha}\int_0^{\mu} d\epsilon f(u) + \sum_{l=1}^{\infty}\frac{(K_B T)^{l+1}}{l!}C_l(\alpha)f^{(l)}(u), \tag{29}$$

where the coefficents are given by

$$C_l(\alpha) = \sum_{k=0}^{l-1}\alpha^k(-1)^{l-k}\left(\begin{array}{c} l \\ k \end{array}\right)\int_0^1 \frac{dw}{w}\ln^k w \ln^{l-k}(1-w). \tag{30}$$

Here $f^{(l)}(u)$ indicates the lth order derivative of the function $f(u)$ with respect to u in the usual way.

From the relation

$$\frac{N}{L} = \frac{1}{2\pi\sqrt{\frac{\hbar^2}{2m}}}\int_0^{\infty} d\epsilon\, n(\epsilon)\epsilon^{-\frac{1}{2}} \tag{31}$$

we find the chemical potential

$$\mu = \mu_0\left[1 + c_2 t^2 + c_3 t^3 + c_4 t^4 + O(t^5)\right] \tag{32}$$

where the coefficients are given by

$$\begin{aligned} c_2 &= \frac{\pi^2\alpha}{12}, \qquad c_3 = -\frac{3}{4}\zeta(3)\alpha(1-\alpha), \\ c_4 &= \frac{\pi^4}{144}\alpha\left(3 - 2\alpha + 3\alpha^2\right), \end{aligned} \tag{33}$$

with $\zeta(3) = \sum_{n=1}^{\infty} 1/n^3$. In the above we introduced an effective temperature $t = K_B T/\mu_0$, where $\mu_0 = \frac{\hbar^2}{2m}k_F^2\alpha^2$ is the anyon cut-off energy at $T = 0$ and $k_F = n\pi$ is the Fermi momentum. The result (32) reduces to the chemical potential for the 1D free Fermi gas in the limit $\gamma \to \infty$.

At finite temperatures, the deviations of the distributions for strongly interacting anyons from Fermi-Dirac statistics are induced by particle excitations which leave holes below the Fermi surface. It is clearly seen that the dynamical interaction γ and the anyonic statistical interaction κ continuously vary the GES, with the most probable distribution of anyons with nonmutual GES approaching free fermion statistics as the interaction increases. Distribution profiles obtained from the GES results (22) and (32) are shown in Figure 1.

Some analytic expressions for the distribution function may be obtained in terms of the variable $x = \mathrm{e}^{\frac{\epsilon-\mu}{K_B T}}$, namely

$$n(\epsilon) = \frac{1}{\alpha}\sum_{m=0}^{\infty} d_m x^{\frac{m}{\alpha}}, \tag{34}$$

which is seen to hold for $\epsilon < 0.9\mu_0$. The coefficients d_m are $d_0 = 1$ and

$$d_m = \frac{(-1)^m}{m!\alpha^m}\prod_{k=0}^{m-1}(m-\alpha k) \tag{35}$$

for $m = 1,\ldots,4$. We expect that this expression may be exact for all m. On the other hand, for $\epsilon > 1.1\mu_0$, we have a universal relation $n(\epsilon) = \sum_{m=1}^{\infty} Q_m x^{-m}$ [18], where

$$Q_m = \prod_{k=1}^{m-1}\left(1-\alpha\frac{m}{k}\right). \tag{36}$$

All distribution curves pass through the inflection point $1/(2\alpha)$. The interactions γ and κ are seen to vary the height of the plateaus. This change of height with respect to γ has been observed in the experimental momentum distribution profiles for the lattice TG gas [14]. For $\kappa = \pi - \theta$ and $\kappa = \pi + \theta$, the anyonic TG gas and the super anyonic TG gas form a GES duality between α and $1/\alpha$, for example, for $\kappa = 0$ and $\kappa = 2\pi$.

On the other hand, for the strong coupling limit, the distribution function obtained from the TBA (15) and (18) is

$$n(\epsilon) = \frac{1}{\alpha(1+\mathrm{e}^{\frac{(\epsilon-\mu)}{K_BT}})} \tag{37}$$

with $\epsilon = \frac{\hbar^2}{2m}k^2$. After some algebra, we find that

$$\begin{aligned} I[f] &= \int_0^\infty d\epsilon f(\epsilon)n(\epsilon) \\ &= \frac{1}{\alpha}\left(\int_0^\mu f(\epsilon)d\epsilon + \frac{\pi^2}{6}(K_BT)^2 f^{(1)}(u)\right. \\ &\quad + \frac{7\pi^4}{360}(K_BT)^4 f^{(3)}(u) + \frac{31\pi^6}{126\times 5!}(K_BT)^6 f^{(5)}(u) \\ &\quad \left. + \frac{127\pi^8}{120\times 7!}(K_BT)^8 f^{(7)}(u) + O((K_BT)^{10})\right). \end{aligned} \tag{38}$$

It follows that

$$\mu = \mu_0\left[1+\frac{\pi^2}{12}t^2+\frac{\pi^4}{36}t^4+O(t^6)\right] \tag{39}$$

with $\mu_0 = \frac{\hbar^2}{2m}k_F^2\alpha^2$. Figure 1 also shows the close agreement between the profile (37) of strongly interacting anyons derived from the TBA and the most probable distribution (22) for ideal particles obeying nonmutual GES. In general both results are seen to coincide well at low temperatures in the strongly interacting regime.

V. THERMODYNAMICS AND LOCAL CORRELATIONS

Thermodynamic properties, such as the pressure P and the total energy E, may be calculated directly from the integral $I[f]$ given in Ref. [18] for ideal particles obeying GES on the one hand and via the TBA result (38) on the other. The pressure and total energy following from the GES result (22) and (29) are

$$\begin{aligned} P &= \frac{2}{3}n\mu_0\left[1+3c_2t^2+2c_3t^3+\frac{9}{5}c_4t^4+O(t^5)\right], \quad (40) \\ E &= E_0\left[1+3c_2t^2+2c_3t^3+\frac{9}{5}c_4t^4+O(t^5)\right]. \quad (41) \end{aligned}$$

Here $E_0 = \frac{1}{3}N\mu_0$ is the ground state energy at zero temperature [11]. On the other hand, the pressure and total energy obtained from the TBA result (38) are

$$\begin{aligned} P &= \frac{2}{3}n\mu_0\left[1+\frac{\pi^2}{4}t^2+\frac{\pi^4}{20}t^4+O(t^6)\right], \quad (42) \\ E &= E_0\left[1+\frac{\pi^2}{4}t^2+\frac{\pi^4}{20}t^4+O(t^6)\right]. \quad (43) \end{aligned}$$

The TBA results coincide with the corresponding GES results (40) and (41) in the TG regime at low temperatures, i.e., for large γ and $T \ll T_d$.

At low temperature $T \ll T_d$, the leading terms in the thermodynamic properties reveal the signature of the degenerate anyon gas. For the super anyonic TG gas with $\pi < \kappa < 3\pi$ the total energy is larger than the free Fermi energy $E_{\mathrm{F}} = \frac{1}{3}N\left(\frac{\hbar^2}{2m}k_F^2\right)$. In particular, the anyonic parameter κ varies the GES parameter from $\alpha < 1$ to

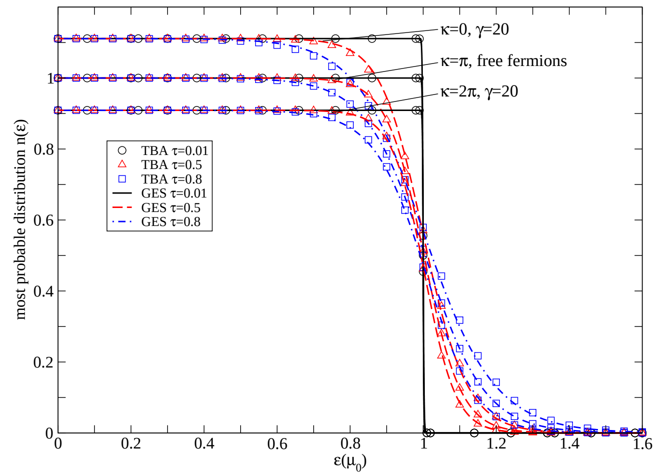

FIG. 1: Comparison between the most probable distribution profiles $n(\epsilon)$ for the values $\gamma = 20$ and $\kappa = 0$, π, 2π at different values of the degeneracy temperature $\tau = K_BT/T_d$. At zero temperature $n(\epsilon) = 1/\alpha$ leads to an anyon surface at $\epsilon = \mu_0$. At finite temperatures these surfaces are gradually diminished by a large number of holes below the surface. Pure Fermi-Dirac statistics appear for $\gamma \to \infty$ or $\kappa = \pi, 3\pi$. For temperature $\tau < 0.1$, thermal fluctuations are suppressed by the dynamical interaction. The lines show the most probable GES distribution derived from (26) with (32). The symbols show the corresponding distributions evaluated from the TBA result (37) for interacting anyons.

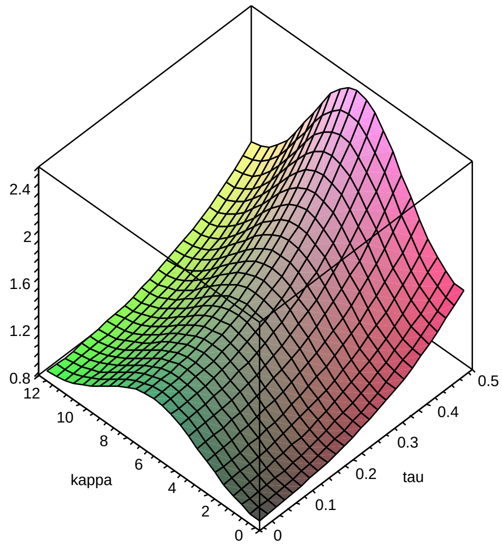

FIG. 2: The energy per particle in units of the free Fermi energy $E_{\rm F}$ vs the effective temperature $\tau = K_BT/T_d$ and the anyonic parameter κ. Here the interaction strength is fixed to be $\gamma = 20$ in (41) in the strong coupling regime. For $0 < \kappa < \pi$ the energy curves interpolate between strongly interacting bosons at $\kappa = 0$ and free fermions at $\kappa = \pi$. For the super TG anyons where $\pi < \kappa < 3\pi$, the energy is greater than the energy for free fermions. For $3\pi < \kappa < 4\pi$ the interpolation is from free fermions to interacting bosons.

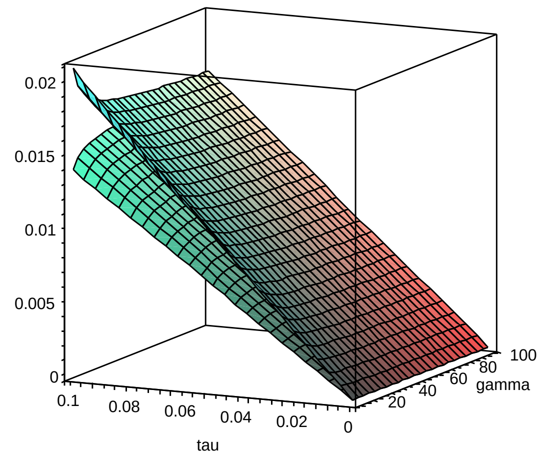

FIG. 3: Specific heat (45) in units of NK_B vs the degeneracy temperature $\tau = K_BT/T_d$ and the interaction parameter γ. The upper layer is the TG Bose gas with $\kappa = 0$, while the lower layer is the super anyonic TG gas with $\kappa = 2\pi$. The specific heat decreases with increasing interaction for the TG Bose gas while it increases with increasing interaction for the super anyonic TG gas. They form a GES duality between α and $1/\alpha$ in the strong coupling limit. For both cases the specific heat is almost linearly increasing with the temperature.

$\alpha > 1$, i.e., it may trigger a phase beyond the free Fermi gas. Fig. 2 shows that the energy (41) increases as the effective temperature increases. We see clearly from Fig. 2 that the anyonic parameter triggers a phase beyond the free Fermi gas for $\pi < \kappa < 3\pi$.

In the grand canonical description, the free energy per particle f follows from $f = F/N = \mu - PL/N$. Thus the specific heat is given by

$$c_v = -T\left(\frac{\partial^2 F}{\partial T^2}\right)_{N,L} = \left(\frac{\partial E}{\partial T}\right)_{N,L} \qquad (44)$$

with result

$$c_v = \frac{NK_B\tau}{6\alpha}\left[1 - \frac{9(1-\alpha)\zeta(3)}{\pi^4\alpha^2}\tau + \frac{(3-2\alpha+3\alpha^2)}{10\pi^2\alpha^4}\tau^2\right]. \qquad (45)$$

For $\gamma \to \infty$, the second term in c_v vanishes, recovering the result for the free Fermi gas, up to irrelevant corrections at low temperatures. Here the TBA gives the result

$$c_v = \frac{NK_B\tau}{6\alpha^2}\left(1 - \frac{4}{10\pi^2\alpha^4}\tau^2\right). \qquad (46)$$

We plot the specific heat in units of NK_B against the degeneracy temperature $\tau = K_BT/T_d$ and interaction strength γ in Fig. 3. The specific heat almost linearly increases as the temperature increases. However, for strongly coupled anyons c_v deviates upwards from the free Fermi curve as γ becomes weaker. This is mainly because the dynamical interaction γ lowers the entropy in fermionization. While for the super anyonic TG gas, c_v deviates downwards from the free Fermi curve as γ decreases. This kind of GES-dependent specific heat is also seen for the three-dimensional ideal anyon gas [9, 25]. In general the specific heat reveals an important signature of quantum statistics for interacting many-body systems.

The local two-particle correlations can be used to classify various finite temperature regimes and study phase coherence behaviour in 1D interacting quantum gases [26]. In a significant advance, the local pair correlations for the 1D Bose gas have been observed experimentally by measuring photoassociation rates [17]. We now consider the local pair correlations for the 1D anyon gas and their role in revealing GES. In the grand canonical description, the two-particle local correlations are given by [26]

$$g^{(2)}(0) = \frac{2m}{\hbar^2 n^2}\left(\frac{\partial f(\gamma,t)}{\partial\gamma}\right)_{n,t}. \qquad (47)$$

Using the GES result (22) we obtain the two-particle lo-

cal correlations at low temperatures

$$g^{(2)}(0) \approx \frac{4\pi^2\cos(\frac{\kappa}{2})}{3\gamma^2}\left(1+\frac{\tau^2}{8\pi^2}-\frac{3\zeta(3)\tau^3}{8\pi^6}+\frac{\tau^4}{480\pi^4}\right) \tag{48}$$

which is to be compared with the TBA result

$$g^{(2)}(0) \approx \frac{4\pi^2\cos(\frac{\kappa}{2})}{3\gamma^2}\left(1+\frac{\tau^2}{4\pi^2}+\frac{3\tau^4}{80\pi^4}\right). \tag{49}$$

For $\kappa = 0$ the latter coincides with the result for the 1D interacting Bose gas [26]. We see that the dynamical interaction dramatically reduces the pair correlations due to decoherence between individual wave functions. The local pair correlations increase with increasing temperature. At temperatures $T \ll T_d$, the local pair correlations approach free Fermi behaviour as $\gamma \to \infty$ or $\kappa \to \pi$. This quantitatively agrees with the experimental observation for 1D interacting bosons [17].

VI. CONCLUSION

To conclude we have demonstrated that the statistical signature of the strongly interacting anyon gas is equivalent at low temperatures to a gas of ideal particles obeying nonmutual GES. Through GES and the TBA, we have independently derived and compared analytic results for the thermodynamics and local pair correlations. Both the dynamical interaction and the anyonic statistical interaction implement a continuous range of GES. They implement a duality between the GES parameter α and $1/\alpha$ and trigger interesting degenerate behaviour at low temperatures. It is shown that the statistical profiles of quasiparticles for the 1D interacting model of anyons are influenced by both the dynamical interaction and anyonic statistical interaction. More generally it will be interesting to see how the statistical properties of degenerate systems are affected by dimensionality, dynamical interaction, topological effects and internal degrees of freedom. In future we hope to address how internal spin degrees affect the statistical profiles of interacting particles [27, 28]. The statistical profiles reveal an important collective signature of interacting quantum degenerate gases. This suggests the possibility of observing the quantum degenerate behaviour of ideal particles obeying GES through experiments on 1D interacting systems.

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